LOGARITHMS AND THEIR PROPERTIES

Definition of a logarithm: If $x > 0$ and b is a constant $(b \neq 1)$, then $y = \log_b x$ if and only if $b^y = x$.
In the equation $y = \log_b x$, y is referred to as the **logarithm,** b is the **base**, and x is the **argument.**

The notation $\log_b x$ is read "the logarithm (or log) base b of x." The definition of a logarithm indicates that *a logarithm is an exponent.*

$y = \log_b x$ is the logarithmic form of $b^y = x$

$b^y = x$ is the exponential form of $y = \log_b x$

Examples of changes between logarithmic and exponential forms:

Write each equation in its exponential form.

a. $2 = \log_7 x$ **b.** $3 = \log_{10}(x + 8)$ **c.** $\log_5 125 = x$

Solution:

Use the definition $y = \log_b x$ if and only if $b^y = x$.

a.

Logarithms are exponents

$2 = \log_7 x$ if and only if $7^2 = x$

Base

b. $3 = \log_{10}(x + 8)$ if and only if $10^3 = (x + 8)$.

c. $\log_5 125 = x$ if and only if $5^x = 125$.

Write the following in its logarithmic form: $x = 25^{1/2}$

Solution:

Use $x = b^y$ if and only if $y = \log_b x$.

Exponent

$$x = 25^{1/2} \text{ if and only if } \frac{1}{2} = \log_{25} x$$

Base

Equality of Exponents Theorem: If b is positive real number $(b \neq 1)$ such that $b^x = b^y$, then $x = y$.

Example of Evaluating a Logarithmic Equation:

Evaluate: $\log_2 32 = x$

Solution:

$\log_2 32 = x$ if and only if $2^x = 32$

Since $32 = 2^5$, we have $2^x = 2^5$

Thus, by Equality of Exponents, $x = 5$

PROPERTIES OF LOGARITHMS:

If b, a, and c are positive real numbers, $b \neq 1$, and n is a real number, then:

1. Product: $\log_b(a \cdot c) = \log_b a + \log_b c$
2. Quotient: $\log_b \frac{a}{c} = \log_b a - \log_b c$
3. Power: $\log_b a^n = n \cdot \log_b a$
4. $\log_b 1 = 0$
5. $\log_b b = 1$
6. Inverse 1: $\log_b b^n = n$
7. Inverse 2: $b^{\log_b n} = n, \ n > 0$
8. One-to-One: $\log_b a = \log_b c$ if and only if $a = c$
9. **Change of Base:** $\log_b a = \frac{\log_c a}{\log_c b} = \frac{\log a}{\log b} = \frac{\ln a}{\ln b}$

Examples – Rewriting Logarithmic Expressions Using Logarithmic Properties:

Use the properties of logarithms to rewrite each expression as a single logarithm:

a. $2\log_b x + \frac{1}{2}\log_b(x+4)$

b. $4\log_b(x+2) - 3\log_b(x-5)$

Solution:

a. $2\log_b x + \frac{1}{2}\log_b(x+4)$

$= \log_b x^2 + \log_b(x+4)^{1/2}$ **Power Property**

$= \log_b\left[x^2(x+4)^{1/2}\right]$ **Product Property**

b. $4\log_b(x+2) - 3\log_b(x-5)$

$= \log_b(x+2)^4 - \log_b(x-5)^3$ **Power Property**

$= \log_b \frac{(x+2)^4}{(x-5)^3}$ **Quotient Property**

Use the properties of logarithms to express the following logarithms in terms of logarithms of x, y, and z.

a. $\log_b(xy^2)$

b. $\log_b \frac{x^2\sqrt{y}}{z^5}$

Solution:

a. $\log_b(xy^2) = \log_b x + \log_b y^2$ **Product Property**

$= \log_b x + 2\log_b y$ **Power Property**

b. $\log_b \frac{x^2\sqrt{y}}{z^5}$

$= \log_b\left(x^2\sqrt{y}\right) - \log_b z^5$ **Quotient Property**

$= \log_b\left(x^2\sqrt{y}\right) - \log_b z^5$ **Quotient Property**

$= \log_b x^2 + \log_b \sqrt{y} - \log_b z^5$ **Product Property**

$= 2\log_b x + \frac{1}{2}\log_b y - 5\log_b z$ **Power Property**

Other Logarithmic Definitions:

- Definition of **Common Logarithm**:

Logarithms with a base of 10 are called **common logarithms.** It is customary to write $\log_{10} x$ as $\log x$.

- Definition of **Natural Logarithm:**

Logarithms with the base of e are called **natural logarithms.** It is customary to write $\log_e x$ as $\ln x$.