*. [Maximum mark: 29]

This question asks you to investigate a sequence of functions given by $f_n(x) = \sum_{k=1}^{n} (|x+k| - |x-k|)$.

Consider the function $f_1(x) = |x+1| - |x-1|$.

(a) Write down the values of $f_1(-1)$, $f_1(0)$, $f_1(1)$. [3]

(b) Show that $f_1(x)$ is an odd function. [3]

(c) Without using a calculator, rewrite the function as a piecewise function, showing your working clearly. [3]

Consider the function $f_2(x) = |x+2| + |x+1| - |x-1| - |x-2|$.

(d) Show that $f_2(x)$ is an odd function. [2]

(e) Find the equations of the line segments joining

(i) the points $(-2, f_2(-2))$ and $(-1, f_2(-1))$;

(ii) the points $(-1, f_2(-1))$ and $(1, f_2(1))$;

(iii) the points $(1, f_2(1))$ and $(2, f_2(2))$. [6]

(f) With the help of technology, draw the graphs of $f_1(x)$ and $f_2(x)$ on the same set of axes. [3]

As seen from part (f), the graphs of $f_1(x)$ and $f_2(x)$ are composed of a pair of horizontal lines and one or more line segments. Explore the graphs of $f_n(x) = \sum_{k=1}^{n} (|x+k| - |x-k|)$ with your graphic display calculator by varying the value of n.

(g) Copy and complete the following table for $n = 1, 2, 3, 4$. [2]

n	The vertical distance between the two horizontal lines of the graph of $y = f_n(x)$
1	
2	
3	
4	

(h) Suggest an expression, in terms of n, the vertical distance between the two horizontal lines of the graph of $y = f_n(x)$. [2]

(i) Using the results of parts (b) and (d), make a conjecture about $f_n(x)$. [1]

(j) Prove the conjecture in part (i). [3]

(k) Using the results of parts (c) and (e)(ii), express the equation of the line segment joining the points $(-1, f_n(-1))$ and $(1, f_n(1))$ in terms of n. [1]

Markscheme

*. (a) $f_1(-1)=-2$, $f_1(0)=0$, $f_1(1)=2$ A1 A1 A1

(b) $f_1(x)=|x+1|-|x-1|$

$f_1(-x)=|-x+1|-|-x-1|$ A1

$=|-(x-1)|-|-(x+1)|$ A1

$=|x-1|-|x+1|$

$=-f_1(x)$ A1

$\therefore$ $f_1(x)$ is an odd function. AG

(c) $f_1(x)=|x+1|-|x-1|$

For $x<-1$, $f_1(x)=-(x+1)-[-(x-1)]$

$=-x-1+x-1$

$=-2$

M1 Consider three cases

A1 Correct removal of absolute signs

For $-1\le x\le 1$, $f_1(x)=(x+1)-[-(x-1)]$

$=x+1+x-1$

$=2x$

For $x>1$, $f_1(x)=(x+1)-(x-1)$

$=2$

$$\therefore\ f_1(x)=\begin{cases}-2, & x<-1\\ 2x, & -1\le x\le 1\\ 2, & x>1\end{cases}$$ A1

(e) $f_2(-x)=|-x+2|+|-x+1|-|-x-1|-|-x-2|$

$=|-(x-2)|+|-(x-1)|-|-(x+1)|-|-(x+2)|$ A1

$=|x-2|+|x-1|-|x+1|-|x+2|$ A1

$=-f_2(x)$

$\therefore$ $f_2(x)$ is an odd function. AG

(d) (i) The equation of the line segment joining $(-2, f_2(-2))$ and $(-1, f_2(-1))$, that is, (–2, –6) and (–1, –4), is

$$y-(-4)=\frac{-4-(-6)}{-1-(-2)}[x-(-1)]$$ A1

$y+4=2(x+1)$

$y=2x-2$ A1

(ii) The equation of the line segment joining $(-1, f_2(-1))$ and $(1, f_2(1))$, that is, (–1, –4) and (1, 4), is

$$y-4=\frac{4-(-4)}{1-(-1)}(x-1)$$ A1

$y-4=4(x-1)$

$y=4x$ A1

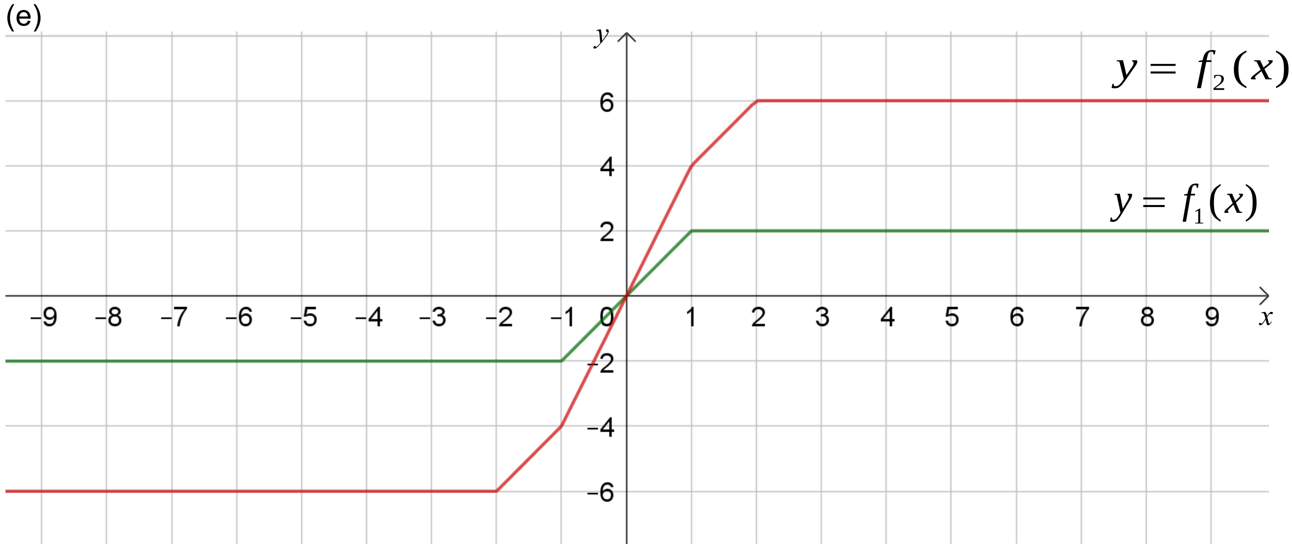
(iii) The equation of the line segment joining $(1, f_2(1))$ and $(2, f_2(2))$, that is, (1, 4) and (2, 6), is

$y-4=\frac{6-4}{2-1}(x-1)$ A1

$y-4=2(x-1)$

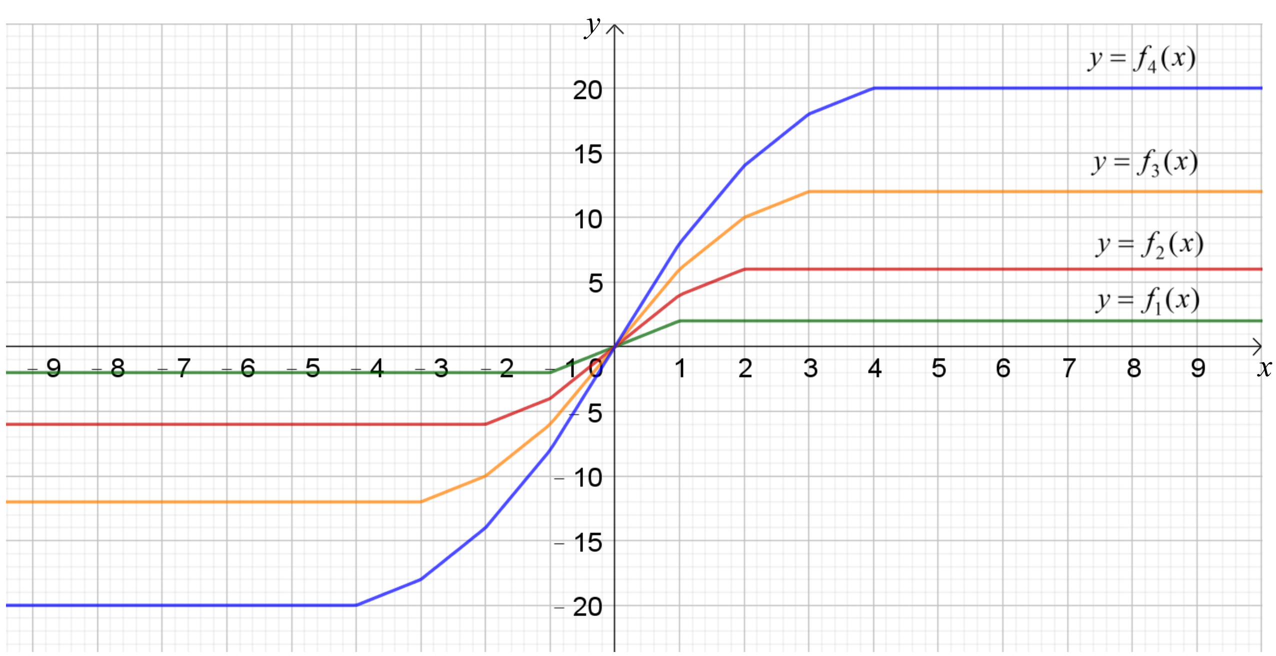
$y=2x+2$ A1

(e)



Graph of $f_1(x)$ A1

Graph of $f_2(x)$ A1 (for horizontal parts) A1 (for 3 line segments)



(f)

n	The vertical distance between the two horizontal lines of the graph of $y = f_n(x)$
1	$4 = 2(2) = 2(1)(2)$
2	$12 = 2(6) = 2(2)(3)$
3	$24 = 2(12) = 2(3)(4)$
4	$40 = 2(20) = 2(4)(5)$

A1 for the first two answers
A1 for the remaining answers

(g) The vertical distance between the two horizontal lines of the graph of $y = f_n(x)$ is $2n(n+1)$, where $n \in \mathbb{Z}^+$. **A2**
(or any equivalent form)

(h) $f_n(x)$ is an odd function where $n \in \mathbb{Z}^+$. **A1**

(i)
$$f_n(-x) = \sum_{k=1}^{n} (|-x+k| - |-x-k|)$$
$$= \sum_{k=1}^{n} (|-(x-k)| - |-(x+k)|) \quad \textbf{A1}$$
$$= \sum_{k=1}^{n} (|x-k| - |x+k|) \quad \textbf{A1}$$
$$= -\sum_{k=1}^{n} (|x+k| - |x-k|) \quad \textbf{A1}$$
$$= -f_n(x)$$

$\therefore$ $f_n(x)$ is an odd function for all $n \in \mathbb{Z}^+$. **AG**

(j) Using the results of parts (c) and (e)(ii), the equation of the line segment joining the points $(-1, f_n(-1))$ and $(1, f_n(1))$ is $y = 2nx$. **A1**