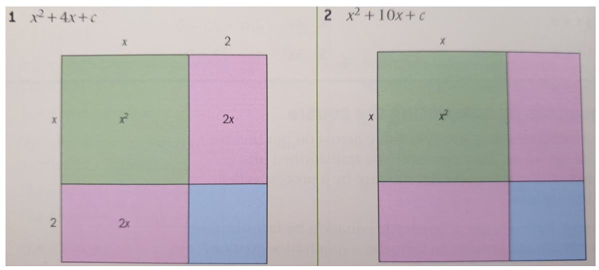
Introduction to complex numbers

Mathematics: Analysis and Approaches HL

November 21, 2021

1. Consider the quadratic expression $A = ax^2 + bx + c$, which represents the area of a square. For the diagrams below, determine the value of c that will make the quadratic expression a perfect square:



2. For the following quadratics, find the value of k that will make the quadratic a perfect square:

- $x^2 + 8x + k$; $k =$ _________
- $x^2 - 7x + k$; $k =$ _________
- $9x^2 + 3x + k$; $k =$ _________

3. We can solve quadratic equations by completing the square as in the example below:

$$\begin{aligned} x^2 - 6x + 1 &= 0 \\ x^2 - 6x &= -1 \\ x^2 - 6x + 3^2 &= 1 + 3^2 \\ (x-3)^2 &= 10 \\ x - 3 &= \pm\sqrt{10} \\ \therefore x &= 3 \pm \sqrt{10} \end{aligned} \tag{1}$$

Similarly, if the equation has a leading coefficient $a \neq 1$, for example:

$$
\begin{aligned}
2x^2 + 5x + 1 &= 0 \\
2x^2 + 5x &= -1 \\
2\left(x^2 + \frac{5}{2}x\right) &= -1 \\
x^2 + \frac{5}{2}x &= -\frac{1}{2} \\
x^2 + \frac{5}{2}x + \left(\frac{5}{4}\right)^2 &= -\frac{1}{2} + \left(\frac{5}{4}\right)^2 \\
\left(x - \frac{5}{4}\right)^2 &= \frac{17}{16} \\
x - \frac{5}{4} &= \pm\sqrt{\frac{17}{16}} \\
x &= \frac{5}{4} \pm \sqrt{\frac{17}{16}} \\
\therefore x &= \frac{5 \pm \sqrt{17}}{4}
\end{aligned}
\tag{2}
$$

Solve the following quadratic equations using the method of completing the square:

- $x^2 + 5x - 24 = 0$
- $2x^2 - 7x + 1 = 0$

4. Use the method of completing the square to solve the general equation $ax^2 + bx + c = 0$.

5. A quadratic equation ALWAYS has two solutions. According to your solution in the previous question, answer the following:

   - Without solving the equation, what condition is needed to have 2 real solutions?
   - Without solving the equation, what condition is needed to have 1 real (repeated) solution?
   - Without solving the equation, what condition is needed to have 0 real solutions?
   - The general solution

     $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

     is known as the quadratic formula and can be used to find the solution(s) of any quadratic equation. Using this formula, find the solutions of the quadratic equations below:
     - $x^2 + 11x + 24 = 0$
     - $3x^2 + 8x + 4 = 0$
     - $2x^2 + x + 1 = 0$

6. Without solving, determine the nature of the roots of each equation:

   - $5x^2 + 7x - 1 = 0$
   - $25x^2 + 49 = 70x$
   - $2x^2 + \frac{7}{4}x + \frac{1}{2} = 0$

7. Find the value of p so that the equation $2x^2 - 3x + p = 0$ has no real solutions.

8. Find the value of p so that the equation $px^2 + p = 13x$ has one real root.

9. Find the value of r so that the equation $x^2 + 3rx + 1 = 0$ has $i)$ two real roots or $ii)$ One repeated root or $iii)$ no real roots.

10. Consider the quadratic equation $x^2 - 4 = 0$. This equation has two real solutions, $x = 2$ and $x = -2$. Can you find real solutions for the equation $x^2 + 4 = 0$? Explain.

11. Consider the quadratic equation $i^2 = -1$. Solve for i.

12. the imaginary number i has been defined as the solution of the quadratic equation $i^2 = -1$, and is usually introduced as $i = \sqrt{-1}$. Using this new number i and its definition, give the solutions to the quadratic equation $2x^2 + x + 1 = 0$ in terms of i.

13. Find the solutions of the equation $x^2 - 2x + 5 = 0$.