Investigation – Roots of complex numbers

1. For $z^n = w$, there should be n solutions.

2. $z = \rho \operatorname{Cis}\alpha$, $w = r \operatorname{cis}\theta$.

$(\rho \operatorname{Cis}\alpha)^n = r \operatorname{cis}\theta$

$\rho^n \operatorname{Cis} n\alpha = r \operatorname{cis}\theta$

$\therefore \quad \rho^n = r \Rightarrow \boxed{\rho = r^{1/n}}$

$n\alpha = \theta + 2\pi k \Rightarrow \boxed{\alpha = \frac{\theta + 2\pi k}{n}}$

3. $z = r^{1/n} \operatorname{Cis}\left(\frac{\theta + 2\pi k}{n}\right)$; $\quad k = 0, 1, \dots, n-1.$

4. $z^3 = 1 \rightarrow z_1 = 1$, $z_2 = \operatorname{Cis}\left(\frac{2\pi}{3}\right)$, $z_3 = \operatorname{Cis}\left(-\frac{2\pi}{3}\right)$

$z^4 = 1 \rightarrow z_1 = 1$, $z_2 = -1$, $z_3 = i$, $z_4 = -i$

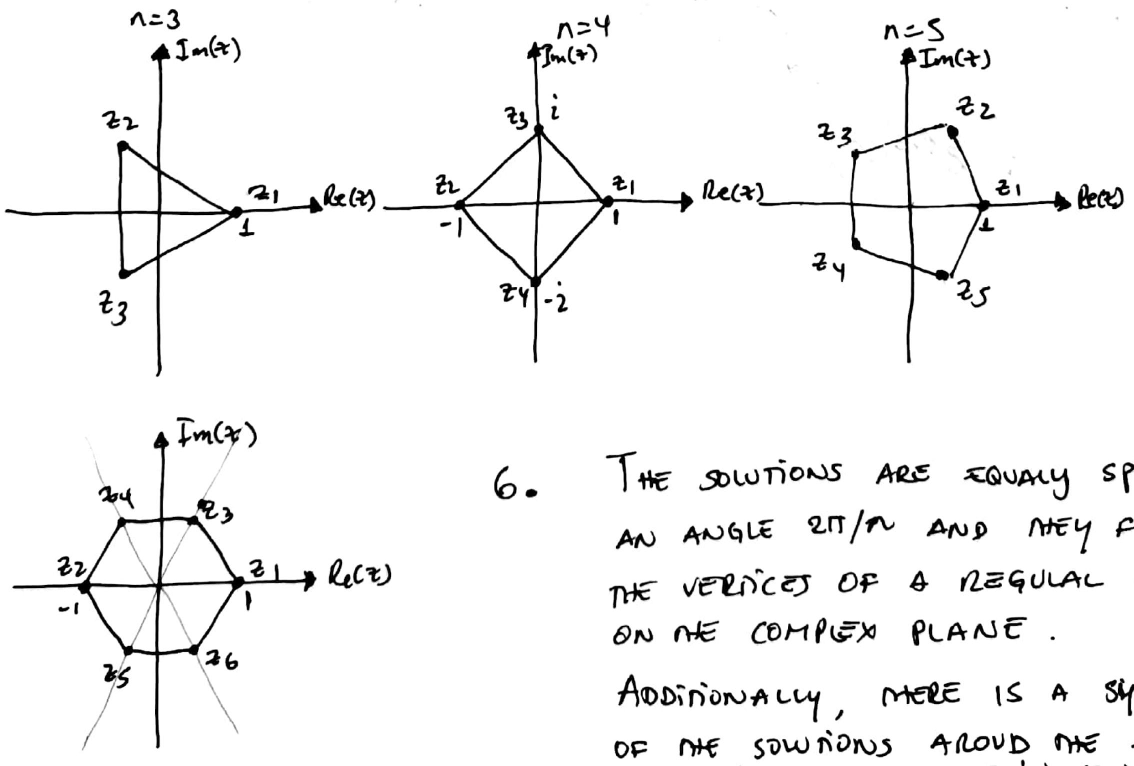
$z^5 = 1 \rightarrow z_1 = 1$, $z_2 = \operatorname{Cis}\left(\frac{2\pi}{5}\right)$, $z_3 = \operatorname{Cis}\left(\frac{4\pi}{5}\right)$

$z_4 = \operatorname{Cis}\left(-\frac{4\pi}{5}\right)$, $z_5 = \operatorname{Cis}\left(-\frac{2\pi}{5}\right)$

$z^6 = 1 \rightarrow z_1 = 1$, $z_2 = -1$, $z_3 = \operatorname{Cis}\left(\frac{\pi}{3}\right)$, $z_4 = \operatorname{Cis}\left(\frac{2\pi}{3}\right)$

$z_5 = \operatorname{Cis}\left(-\frac{2\pi}{3}\right)$, $z_6 = \operatorname{Cis}\left(-\frac{\pi}{3}\right)$

5.



6. THE SOLUTIONS ARE EQUALLY SPACED AN ANGLE $2\pi/n$ AND THEY FORM THE VERTICES OF A REGULAR POLYGON ON THE COMPLEX PLANE.

ADDITIONALLY, THERE IS A SYMMETRY OF THE SOLUTIONS AROUND THE HORIZONTAL AXIS. (SOLUTIONS COME IN CONJUGATE PAIRS).

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7. $z^6 = 125e^{i\frac{2\pi}{3}}$

$\therefore \quad z = \sqrt[6]{125}\, Cis\left(\dfrac{\frac{2\pi}{3} + 2\pi k}{6}\right), \quad k = 0, 1, 2, 3, 4, 5.$

$z_1 = \sqrt{5}\, Cis\left(\frac{\pi}{9}\right), \quad z_2 = \sqrt{5}\, Cis\left(\frac{4}{9}\pi\right)$

$z_3 = \sqrt{5}\, Cis\left(\frac{7\pi}{9}\right), \quad z_4 = \sqrt{5}\, Cis\left(-\frac{2\pi}{9}\right)$

$z_5 = \sqrt{5}\, Cis\left(-\frac{5\pi}{9}\right), \quad z_6 = \sqrt{5}\, Cis\left(-\frac{8\pi}{9}\right)$

8. THIS WAS DONE IN 4:

$z_1 = 1, \quad z_2 = -1, \quad z_3 = Cis\left(\frac{\pi}{3}\right), \quad z_4 = Cis\left(\frac{2\pi}{3}\right), \quad z_5 = Cis\left(-\frac{2\pi}{3}\right), \quad z_5 = Cis\left(-\frac{\pi}{3}\right)$

9.

ROOTS OF UNITY	ROOTS OF $125e^{i2\pi/3}$
$1 = e^{0i}$	$\sqrt{5}\, e^{\pi/9\, i}$
$-1 = e^{\pi i}$	$\sqrt{5}\, e^{-8\pi/9\, i}$
$e^{\pi/3\, i}$	$\sqrt{5}\, e^{4\pi/9\, i}$
$e^{\frac{2\pi}{3} i}$	$\sqrt{5}\, e^{\frac{7\pi}{9} i}$
$e^{-2\pi/3\, i}$	$\sqrt{5}\, e^{-\frac{5\pi}{9} i}$
$e^{-\pi/3\, i}$	$\sqrt{5}\, e^{-2\pi/9\, i}$

$\times \sqrt{5}\, e^{\pi/9\, i}$

WE CAN GET THE ROOTS OF $125e^{i2\pi/3}$ BY MULTIPLYING THE ROOTS OF UNITY BY $\sqrt{5}e^{\frac{\pi}{9}i}$, WHICH MEANS, A ROTATION OF $\frac{\pi}{9}$ COUNTERCLOCKWISE AND A MAGNIFICATION BY $\sqrt{5}$.