Investigation: Fundamental theorem of Calculus

Carolina Buitrago

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Consider the function

$$A(x) = \int_a^x f(t)dt$$

1. Write down an expression for $A(x+h)$.
2. Hence, write down an expression for $A'(x)$ from first principles.
3. Using the properties of the definite integral rewrite the derivative from the previous question, in terms of a definite integral.
4. Given that the average value of a function over an interval $[a,b]$, f_{av} will always be such that $f_{min} \leq f_{av} \leq f_{max}$, where $f_{av} = \frac{1}{b-a}\int_a^b f(x)dx$, evaluate your limit from 3.
5. In 4. you found that $A'(x) = \frac{d}{dx}A(x) = \frac{d}{dx}\int_a^x f(t)dt = f(x)$. This result is known as the first fundamental theorem of Calculus. Using this newly acquired knowledge, evaluate the following expressions:
   (a) $\frac{d}{dx}\int_{-e}^{x} \sec^2 t dt$
   (b) $\frac{d}{dx}\int_0^x \frac{1}{1+t^4}dt$
   (c) $\frac{d}{dx}\int_x^\pi \frac{1}{1+t^4}dt$
   (d) $\frac{d}{dx}\int_0^{2x+x^3} \frac{1}{1+t^4}dt$
   (e) $\frac{d}{dx}\int_x^{2x+x^3} \frac{1}{1+t^4}dt$
6. Consider the antiderivative of the function $f(x)$, $F(x)$. Write an expression showing the relationship between $F(x)$ and $A(x)$, using $F(x)$ as the leading value.
7. Using your expression in 6. write down $F(b)$ and $F(a)$, where $a, b \in \mathbf{R}$.
8. Substituting the corresponding integral and using the properties of the definite integral, show that $F(b) - F(a) = \int_a^b f(t)dt$.

9. The result in 8. is known as the Second fundamental Theorem of Calculus, and it shows that the definite integral can be calculated by taking the difference of the antiderivative evaluated in the upper boundary and its value when evaluated in the lower boundary. Using this theorem, find the area under the following curves:

   (a) x^5, $x \in [-1, 3]$
   (b) $\sqrt{x}$, $x \in [0, 4]$
   (c) $\cos\theta$, $\theta \in [\pi, 2\pi]$.
   (d) $\frac{4+u^2}{u^3}$, $u \in [1, 2]$

10. Express the fraction $\frac{5x+1}{x^2+x-2}$ in partial fractions of the form $\frac{A}{x-1} + \frac{B}{x+2}$.

11. Evaluate the following integral

$$A = \int_2^5 \frac{5x+1}{x^2+x-2} dx$$

12. Evaluate the following integral

$$\int_1^2 \frac{2}{x^3+x}$$