7. What degree Maclaurin polynomial of e^x must be taken to guarantee an estimate of e to within 1×10^{-6}? *[6 marks]*

8. Show that the Maclaurin series for $\sin 2x$ converges to the function for all $x\in\mathbb{R}$. *[8 marks]*

9. (a) Use the Lagrange form of the error term to bound the error involved in approximating $-\ln(1-x)$ at $x=0.5$ by $x+\frac{x^2}{2}+\frac{x^3}{3}$.

(b) By noting that $R_3(0.5)=\frac{(0.5)^4}{4}+\frac{(0.5)^5}{5}+\cdots$ give an improved bound on the error. *[9 marks]*

10. (a) Find the first three terms of the Maclaurin Series for $(1+x)^n$.

(b) How many terms of the expansion of $(1+x)^{82}$ are needed to guarantee finding a value of 1.1^{82} accurate to within 10^{-6}? *[10 marks]*

11. Show that the Maclaurin series for $(1+x)^{-\frac{3}{2}}$ converges to the function for $|x|<1$. *[10 marks]*

4C Maclaurin series of composite functions

The following standard Maclaurin series (which we have already met), appear in the Formula booklet:

KEY POINT 4.5

$$e^x=1+x+\frac{x^2}{2!}+\cdots$$
$$\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\cdots$$
$$\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots$$
$$\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots$$
$$\arctan x=x-\frac{x^3}{3}+\frac{x^5}{5}-\cdots$$

These can be used to find Maclaurin series of more complicated functions. Sometimes this is straightforward.

Worked example 4.6

Using the Maclaurin series for cos x, find the series expansion of $\cos(2x^3)$.

We just need to substitute $2x^3$ into the known series for cos x

$$\cos(2x^3)=1-\frac{(2x^3)^2}{2!}+\frac{(2x^3)^4}{4!}-\frac{(2x^3)^6}{6!}+\cdots$$
$$=1-2x^6+\frac{2}{3}x^{12}-\frac{4}{45}x^{18}+\cdots$$

Often this will involve finding two separate Maclaurin series and then combining them.

Worked example 4.7

Using the Maclaurin series for $\sin x$ and e^x, find the series expansion of $e^{\sin x}$ as far as the term in x^4.

We start by substituting the series for $\sin x$, only going as far as the x^4 term

$$e^{\sin x} = e^{x-\frac{x^3}{3!}+\dots}$$

$$\approx e^x e^{-\frac{x^3}{3!}}$$

We now use the series for e^x only going as far as x^4 and then expand

$$\approx \left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots\right)\left(1+\left(-\frac{x^3}{3!}\right)+\cdots\right)$$

$$\approx 1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}-\frac{x^3}{6}-\frac{x^4}{6}+\cdots$$

$$\approx 1+x+\frac{x^2}{2}-\frac{x^4}{8}$$

EXAM HINT

It is much quicker to form Maclaurin series in this way so where possible in the exam combine known series.

We can also use results on intervals where the expansion converges to the function (as we know these will be the intervals of convergence of the functions we will meet) to find the values for which the expansion of composite functions are valid.

Worked example 4.8

Find the Maclaurin series up to the term in x^3 for $\ln\left(\frac{\sqrt{1+2x}}{2-3x}\right)$ and state the interval in which the expansion is valid.

We know the series expansion for $\ln(1+x)$ so rearrange the original function into separate functions in this form

$$\ln\left(\frac{\sqrt{1+2x}}{2-3x}\right) = \ln\left(\sqrt{1+2x}\right)-\ln(2-3x)$$

$$= \frac{1}{2}\ln(1+2x)-\ln(2-3x)$$

$$= \frac{1}{2}\ln(1+2x)-\ln\left[2\left(1+\left(\frac{-3x}{2}\right)\right)\right]$$

$$= \frac{1}{2}\ln(1+2x)-\left\{\ln 2+\ln\left(1+\left(\frac{-3x}{2}\right)\right)\right\}$$

Now find the series expansion for each separately

$$\frac{1}{2}\ln(1+2x) = \frac{1}{2}\left((2x)-\frac{(2x)^2}{2}+\frac{(2x)^3}{3}+\cdots\right)$$

$$= \frac{1}{2}\left(2x-2x^2+\frac{8x^3}{3}+\cdots\right)$$

→

continued . . .

$$= x - x^2 + \frac{4x^3}{3} + \cdots$$

$$\ln\left(1+\left(\frac{-3x}{2}\right)\right) = \left(\frac{-3x}{2}\right) - \frac{\left(\frac{-3x}{2}\right)^2}{2} + \frac{\left(\frac{-3x}{2}\right)^3}{3} + \cdots$$

$$= \frac{-3x}{2} - \frac{9x^2}{8} - \frac{9x^3}{8} + \cdots$$

And finally put everything together

$$\ln\left(\frac{\sqrt{1+2x}}{2-3x}\right) = \left(x - x^2 + \frac{4x^3}{3} + \cdots\right) - \ln 2 - \left(\frac{-3x}{2} - \frac{9x^2}{8} - \frac{9x^3}{8} + \cdots\right)$$

$$= x - x^2 + \frac{4x^3}{3} + \cdots - \ln 2 + \frac{3x}{2} + \frac{9x^2}{8} + \frac{9x^3}{8} + \cdots$$

$$= \ln\left(\frac{1}{2}\right) + \frac{5x}{2} + \frac{x^2}{8} + \frac{59x^3}{24} + \cdots$$

We consider the interval of validity separately for each function and then note that for both to be valid we need the smaller interval

Since $\ln(1+x)$ is valid when $-1 < x \le 1$,

$\ln(1+2x)$ is valid when $-1 < 2x \le 1$

i.e. when $-\frac{1}{2} < x \le \frac{1}{2}$

$\ln\left(1+\left(\frac{-3x}{2}\right)\right)$ is valid when $-1 < \frac{-3x}{2} \le 1$

i.e. when $-\frac{2}{3} \le x < \frac{2}{3}$

Therefore, $\ln\left(\frac{\sqrt{1+2x}}{2-3x}\right)$ is valid when $-\frac{1}{2} \le x < \frac{1}{2}$

Exercise 4C

In this exercise you can assume all the standard Maclaurin series results given in the Formula booklet.

1. Find the first four non-zero terms of the Maclaurin series for:
   (a) (i) $\sin(3x^4)$ (ii) $\cos\left(2\sqrt{x}\right)$
   (b) (i) $\ln(2+3x)$ (ii) $\ln(1-2x)$
   (c) (i) $e^{-\frac{x^2}{2}}$ (ii) e^{x^3}

2. By combining Maclaurin series of different functions find the series expansion as far as the term in x^4 for:
   (a) (i) $\ln(1+x)\sin 2x$ (ii) $\ln(1-x)\cos 3x$
   (b) (i) $\frac{e^x}{1+x}$ (ii) $\frac{\sin x}{1-2x}$
   (c) (i) $\ln(1+\sin x)$ (ii) $\ln(1-\sin x)$

3. Find the Maclaurin series as far as the term in x^4 for $e^{3x}\sin 2x$. *[4 marks]*

4. Show that $\sqrt{1+x^2}e^{-x} \approx 1 - x + x^2 - \frac{2}{3}x^3 + \frac{1}{6}x^4$. *[5 marks]*

5. (a) Find the Maclaurin series for $\ln(1+4x^2+4x)$, giving your answer in the form $\sum a_k x^k$

(b) State the interval of convergence of the power series.

[6 marks]

6. (a) Find the first two non-zero terms of the Maclaurin series for $\tan x$.

(b) Hence find the Maclaurin series of $e^{\tan x}$ up to and including the term in x^4. *[6 marks]*

7. (a) By using the Maclaurin series for $\cos x$, find the series expansion for $\ln(\cos x)$ up to the term in x^4.

(b) Hence find the first two non-zero terms of the expansion of $\ln(\sec x)$ stating where the expansion is valid.

(c) Use your result from (b) to find the first two non-zero terms of the series for $\tan x$. *[8 marks]*

8. (a) Find the first 4 terms of the Maclaurin series for

$$f(x) = \ln[(2+x)^3(1-3x)]$$

(b) Find the equation of the tangent to $f(x)$ at $x = 0$.

[10 marks]

9. (a) Find the Maclaurin series for $\ln\sqrt{\frac{1+x}{1-x}}$ stating the interval of convergence of the power series.

(b) Use the first three terms of this series to estimate the value of ln 2, stating the value of x used.

(c) Provide an upper bound on the error in your approximation using the Lagrange error term.

(d) Refine the upper bound on the error by considering the error as a geometric series.

10. Using the standard result for e^x, form a series for $e^{\sqrt{x}}$. Why is this not a valid expansion? Does $e^{\sqrt{x}}$ have a Maclaurin series?

[13 marks]

4D Taylor series

We have seen that a Maclaurin series is valid in an interval centred on $x = 0$ and that close to this point, often just a few terms of the series are needed to give a very good approximation of the function. However, further away from $x = 0$, even within the interval where the expansion is valid, you can need many more terms of the series to get a reasonable degree of accuracy.

For example, we found above that the 5th degree Maclaurin polynomial for $\sin x$ approximated sin 0.5 correct to 5DP but