# Paper 3

**Time allowed: 1 hour**

**Maximum number of marks: 60 marks**

Answer all the questions.

All numerical answers must be given exactly or correct to three significant figures, unless otherwise stated in the question.

Answers should be supported by working and/or explanations. Where an answer is incorrect, some marks may be awarded for a correct method, provided this is shown clearly.

**You need a graphic display calculator for this paper.**

**1** [Maximum mark: 26]

In this question, you will attempt to find information about the sums of sequences that are neither arithmetic nor geometric progressions.

**a i** Let $S_n = \sum_{k=1}^{n} \frac{1}{k(k+1)}$ .

Use partial fractions to write $\frac{1}{k(k+1)}$ as the sum of two fractions, both with linear denominators. Hence show that $S_n = \frac{n}{n+1}$ .

**ii** Let $S = \lim_{n\to\infty} S_n$ . Write down the value of $S$. [8]

**b i** Let $T_n = \sum_{k=1}^{n} \frac{1}{k(k+2)}$ . Use a similar method to that in part (a) to find and simplify an expression for $T_n$ .

**ii** Let $T = \lim_{n\to\infty} T_n$ . Write down the value of $T$. [10]

**c i** Let $Q_n = \sum_{k=1}^{n} \frac{1}{k^2}$ . Explain why $\sum_{i=2}^{n} \frac{1}{i^2} < \sum_{i=2}^{n} \frac{1}{(i-1)i}$ and hence show that

$$Q_n < 1 + S_{n-1} = \frac{2n-1}{n}$$

**ii** Let $Q = \lim_{n\to\infty} Q_n$ . Show that $\frac{49}{36} < Q \le 2$ . [8]

**2** [Maximum mark: 34]

In this question, you will investigate the conditions required for a volume of revolution to have the same volume when the curve is rotated about either the $x$-axis or the $y$-axis.

Consider the portion of the line $y = mx$, $m > 0$, that lies in the first quadrant and goes from point $(a, ma)$ to the point $(X, mX)$, $0 \leq a < X$, where $a$ is a constant and $X$ is a variable.

A solid of revolution is obtained by rotating this line through $2\pi$ radians about the $x$-axis and another solid of revolution is obtained by rotating this line through $2\pi$ radians about the $y$-axis.

**a** If the two volumes are the same for all values of $X$, use integration to find the value of $m$. [13]

Now working more generally, consider the portion of the increasing function $y = y(x)$, that lies in the first quadrant and goes from point $(a, b)$ to the point $(X, Y)$, $0 \leq a < X, 0 \leq b < Y$. Here $a$ and $b$ are constants and $X$ and $Y$ are variables. A solid of revolution is obtained by rotating this curve through $2\pi$ radians about the $x$-axis and another solid of revolution is obtained by rotating this curve through $2\pi$ radians about the $y$-axis. If the two volumes are the same for all values of $X$, it can be proved that $X$ and $Y$ must satisfy the differential equation $Y^2 = X^2 \frac{dY}{dX}$

**b** Solve this differential equation to show that $Y = \frac{X}{1 + cX}$ [6]

It follows that the function $y(x) = \frac{x}{1 + cx}$ between $(a, b)$ and $(X, Y)$ produces a volume of revolution of same magnitude, regardless of which axis it is rotated about.

**c** Find $c$ in terms of $a$ and $b$ in the function given above. [3]

**d** Show that the special case $c = 0$ corresponds to the example found in part **a**. [2]

**e** Use calculus to verify that $y(x) = \frac{x}{1 + cx}$ is indeed an increasing function. [3]

**f** Investigate the concavity of $y(x) = \frac{x}{1 + cx}$ and the existence of any asymptotes. Consider the necessary different cases, which depend on the value of $c$. **Remember:** you are only considering the portion of the graph of $y(x) = \frac{x}{1 + cx}$ which lies in the first quadrant, that is $x > 0, y > 0$. [7]

**Markscheme**

**1 a i** $\dfrac{1}{k(k+1)} \equiv \dfrac{A}{k} + \dfrac{B}{k+1} \Rightarrow A(k+1) + Bk \equiv 1 + 0k$ M1 A1

Equating coefficients: $A + B = 0, A = 1 \Rightarrow B = -1$ R1 A1

$S_n = \sum_{k=1}^{n}\left(\dfrac{1}{k} - \dfrac{1}{k+1}\right) = \left(\dfrac{1}{1} - \dfrac{1}{2}\right) + \left(\dfrac{1}{2} - \dfrac{1}{3}\right) + \left(\dfrac{1}{3} - \dfrac{1}{4}\right) + \ldots + \left(\dfrac{1}{n} - \dfrac{1}{n+1}\right)$ M1

$= \left(\dfrac{1}{1} - \dfrac{1}{2}\right) + \left(\dfrac{1}{2} - \dfrac{1}{3}\right) + \left(\dfrac{1}{3} - \dfrac{1}{4}\right) + \ldots + \left(\dfrac{1}{n} - \dfrac{1}{n+1}\right)$

$= 1 + \left(\dfrac{1}{2} - \dfrac{1}{2}\right) + \left(\dfrac{1}{3} - \dfrac{1}{3}\right) + \ldots + \left(\dfrac{1}{n} - \dfrac{1}{n}\right) - \dfrac{1}{n+1}$ A1

$= 1 - \dfrac{1}{n+1}$ A1

$= \dfrac{n}{n+1}$ AG

**ii** 1 A1

[8 marks]

**b i** $\dfrac{1}{k(k+2)} \equiv \dfrac{A}{k} + \dfrac{B}{k+2} \Rightarrow A(k+2) + Bk \equiv 1 + 0k$ M1 A1

Equating coefficients: $A + B = 0,\ 2A = 1 \Rightarrow A = \dfrac{1}{2}, B = -\dfrac{1}{2}$ R1 A1

$T_n = \sum_{k=1}^{n}\dfrac{\frac{1}{2}}{k} + \dfrac{\frac{-1}{2}}{k+2} = \left(\dfrac{\frac{1}{2}}{1} - \dfrac{\frac{1}{2}}{3}\right) + \left(\dfrac{\frac{1}{2}}{2} - \dfrac{\frac{1}{2}}{4}\right) + \left(\dfrac{\frac{1}{2}}{3} - \dfrac{\frac{1}{2}}{5}\right) + \left(\dfrac{\frac{1}{2}}{4} - \dfrac{\frac{1}{2}}{6}\right) \ldots + \left(\dfrac{\frac{1}{2}}{n-1} - \dfrac{\frac{1}{2}}{n+1}\right) + \left(\dfrac{\frac{1}{2}}{n} - \dfrac{\frac{1}{2}}{n+2}\right)$

M1 A1

$= \dfrac{1}{2} + \dfrac{1}{4} + \left(\dfrac{1}{6} - \dfrac{1}{6}\right) + \left(\dfrac{1}{8} - \dfrac{1}{8}\right) + \ldots + \left(\dfrac{1}{2n} - \dfrac{1}{2n}\right) - \dfrac{1}{2(n+1)} - \dfrac{1}{2(n+2)}$

$= \dfrac{1}{2} + \dfrac{1}{4} - \dfrac{1}{2(n+1)} - \dfrac{1}{2(n+2)} = \dfrac{3(n+1)(n+2) - 2(n+2) - 2(n+1)}{4(n+1)(n+2)}$ A1 M1

$= \dfrac{3n^2 + 5n}{4(n+1)(n+2)}$ A1

**ii** $\dfrac{3}{4}$ A1

[10 marks]

**c i** $i > i - 1 \Rightarrow \dfrac{1}{i^2} < \dfrac{1}{(i-1)i} \Rightarrow \sum_{i=2}^{n}\dfrac{1}{i^2} < \sum_{i=2}^{n}\dfrac{1}{(i-1)i}$ R1 R1

$\sum_{i=1}^{n}\dfrac{1}{i^2} < 1 + \sum_{i=2}^{n}\dfrac{1}{(i-1)i} \Rightarrow Q_n < 1 + \sum_{k=1}^{n-1}\dfrac{1}{k(k+1)}$ M1 A1

So $Q_n < 1 + S_{n-1} = 1 + \dfrac{n-1}{n} = \dfrac{2n-1}{n}$ R1 AG

**ii** Addition of the first three terms of $Q_n$ gives $1 + \dfrac{1}{4} + \dfrac{1}{9} = \dfrac{49}{36}$ M1 A1

$\lim_{n\to\infty}\dfrac{2n-1}{n} = 2$ A1

So $\dfrac{49}{36} < Q \le 2$ AG

[8 marks]

[Total: 26 marks]

**2 a** $\pi\int_{a}^{X} y^2 dx = \pi\int_{ma}^{mX} x^2 dy$ M1 A1 A1

$\int_{a}^{X} m^2x^2 dx = \int_{ma}^{mX} \dfrac{y^2}{m^2} dy$ M1 A1 A1

| | | |
|---|---|---|
| $\left[\frac{m^2x^3}{3}\right]_a^X = \left[\frac{y^3}{3m^2}\right]_{ma}^{mX}$ | A1 | A1 |
| $\frac{m^2X^3}{3} - \frac{m^2a^3}{3} = \frac{m^3X^3}{3m^2} - \frac{m^3a^3}{3m^2}$ | A1 | A1 |
| $m^2\left(X^3 - a^3\right) = m\left(X^3 - a^3\right)$ | | A1 |
| $m^2 - m = 0 \Rightarrow m(m-1) = 0 \Rightarrow m = 1$ | M1 | A1 |
| | [13 marks] | |
| **b** $\int \frac{1}{X^2}\,dX = \int \frac{1}{Y^2}\,dY$ | M1 | A1 |
| $\frac{-1}{X} = \frac{-1}{Y} + c$ | A1 | A1 |
| $\frac{1}{Y} = \frac{1}{X} + c = \frac{1+cX}{X}$ | M1 | A1 |
| $Y = \frac{X}{1+cX}$ | | AG |
| | [6 marks] | |
| **c** $b = \frac{a}{1+ca} \Rightarrow b + cab = a \Rightarrow cab = a - b$ | R1 | M1 |
| $c = \frac{a-b}{ab}$ | | A1 |
| | [3 marks] | |
| **d** $c = 0$ gives $y = x$ and $a = b$ which corresponds to part **a** with $m = 1$ | R1 | R1 |
| | [2 marks] | |
| **e** $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(1+cx) - xc}{(1+cx)^2} = \frac{1}{(1+cx)^2}$ | M1 | A1 |
| This is always positive, confirming that the function is increasing. | | R1 |
| | [3 marks] | |
| **f** $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{-2c}{(1+cx)^3}$ | M1 | A1 |
| Since the curve is in the first quadrant, $y, x > 0$ and since $y(x) = \frac{x}{1+cx}$, it follows that | | |
| $1 + cx > 0$ | | R1 |
| If $c > 0$, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} < 0$ and hence $y(x) = \frac{x}{1+cx}$ is concave down, | | R1 |
| with horizontal asymptote $y = \frac{1}{c}$ | | A1 |
| If $c < 0$, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} > 0$ and hence $y(x) = \frac{x}{1+cx}$ is concave up, | | R1 |
| with horizontal asymptote $x = \frac{-1}{c}$ | | A1 |
| If $c = 0$ , it is the straight line given in part **a**. | | |
| | [7 marks] | |
| | [Total: 34 marks] | |