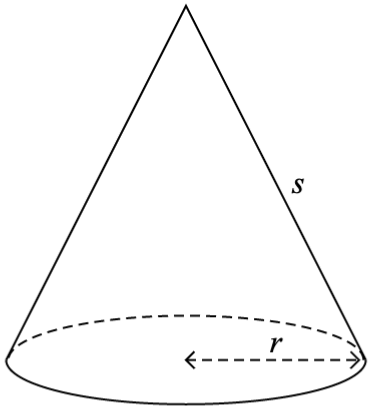
1. [Maximum points: 28]

In this problem you will investigate the area of surfaces generated by rotating functions 360° *around the x-axis.*

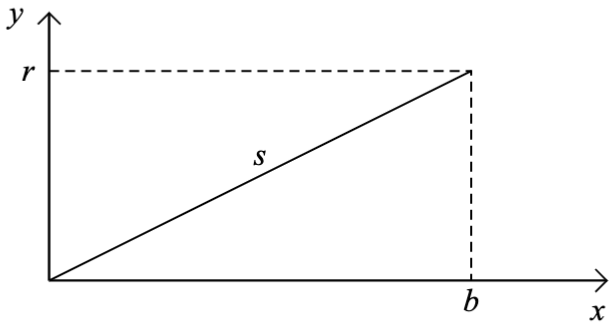
When the function $y = f(x)$ from $x = a$ to $x = b$ is rotated 360° around the x-axis the area A of the surface generated is given by

$$A = 2\pi\int_a^b f(x)\sqrt{1 + (f'(x))^2}\,dx$$

Consider a cone with a slant height of s and base of radius r as shown in the diagram below.



The curved surface is formed when the line with length s in the diagram below from $x = 0$ to $x = b$ is rotated 360° around the x-axis.



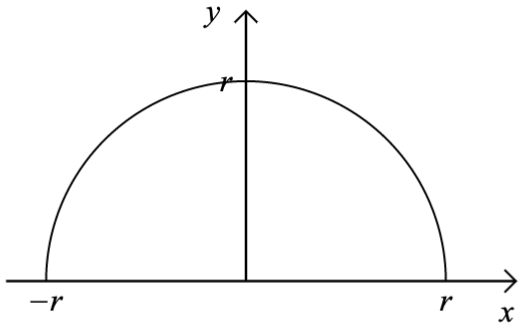
(a) In terms of s and r find [4]

(i) the value of b

(ii) the equation of the line

(b) Hence show that the area of the curved surface of the cone is equal to πrs. [5]

The diagram below shows a semi-circle of radius r centred at the origin. Let the equation of the semi-circle be $y = g(x)$.



(c) Find [5]

(i) $g(x)$

(ii) $g'(x)$

(d) Hence show that the surface area of a sphere is equal to $4\pi r^2$. [6]

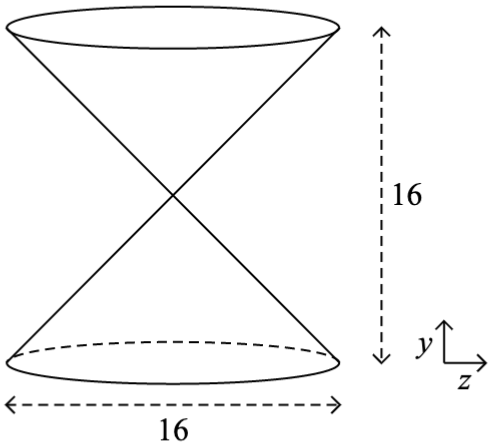
(e) Find the exact area of the surface generated when then function $y = x^3$ from $x = 1$ to $x = 2$ is rotated 360° around the x-axis. [8]

2. [Maximum points: 41]

In this problem you will investigate the shape formed by the intersection of a cone and a plane.

Two identical cones of height 8 with base of diameter 16 placed tip-to-tip with the centre of the top cone directly above the centre of the bottom cone.

This is shown in the diagram below where the directions of the z and y-axes are given, and the positive x-direction is out of the page.



Let the point where the tips meet have coordinates $(0,0,1)$.

Consider a point on the curved surface of a cone with coordinates (x,y,z).

(a) Show that $x^2 + (z-1)^2 = z^2$. [4]

The equation of a **plane** is $z = 0$.

(b) Find the equation of the curve created by the intersection of the plane $z = 0$ and the two cones. Write your answer in the form $ax^2 + by^2 = 1$ where a and b are real numbers to be determined. [2]

(c) For your equation in part (b) find [9]

(i) $\frac{dy}{dx}$

(ii) the coordinates of any turning points

(iii) the equations of any linear oblique asymptotes

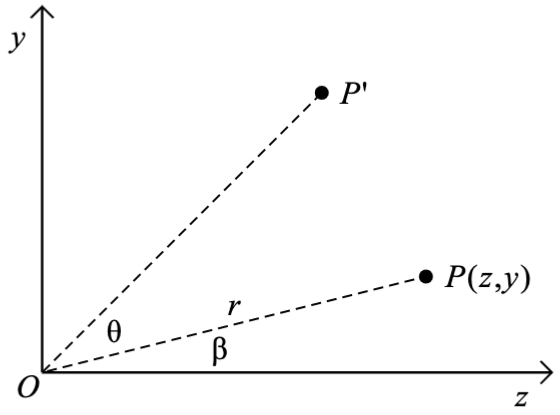
(d) Sketch the curve represented by the equation in part (b) for $-5 \leq x \leq 5$ showing the features from (c) parts (ii) and (iii). [3]

(e) Verify that the vector equation of the curved surface of the cones can be written as [3]

$$\boldsymbol{r} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} \cos t \\ 1 \\ \sin t \end{pmatrix}$$

where $\lambda, t \in \mathbb{R}$.

The diagram below shows point P in the zy-plane with coordinates (z,y) at a distance r from the origin O. The point is rotated by an angle of θ anti-clockwise about the origin to form point P'. The angle between OP and the z-axis is β.



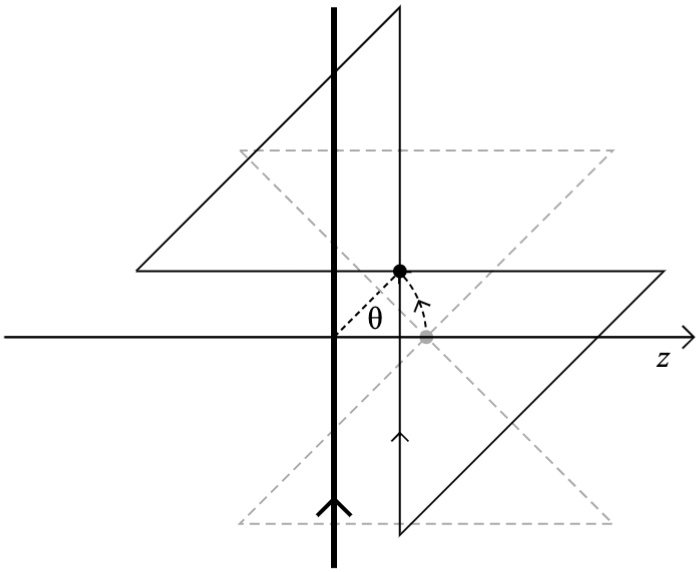
(f) Show that the coordinates of P' are [3]

$$(r\cos\theta\cos\beta - r\sin\theta\sin\beta, r\sin\theta\cos\beta + r\cos\theta\sin\beta)$$

(g) Hence show that this is equal to [2]

$$(z\cos\theta - y\sin\theta, z\sin\theta + y\cos\theta)$$

The cones are now rotated in the zy-plane about the origin so that the plane $z = 0$ is now parallel to one edge of each cone when viewed in the zy-plane. This is shown in the diagram below.



$z = 0$

(h) Calculate the angle of rotation θ. [2]

(i) Show that the vector equation of the curved surface of the rotated cones is [4]

$$\boldsymbol{r} = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\1\end{pmatrix} + \frac{\lambda}{\sqrt{2}}\begin{pmatrix}\sqrt{2}\cos t\\ \sin t + 1\\ \sin t - 1\end{pmatrix}$$

(j) For any point on the curve created by the intersection of the plane $z = 0$ and the rotated cones show that [5]

(i) $y = \lambda\sqrt{2}$

(ii) $x^2 = 2\lambda - 1$

(k) Find the equation of this curve and sketch its graph for $-5 \leq x \leq 5$ showing the coordinates of any axes intercepts. [4]