Mathematics
Higher level
Paper 3 – calculus

Instructions to candidates

- Do not open this examination paper until instructed to do so.
- Answer all the questions.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A graphic display calculator is required for this paper.
- A clean copy of the **mathematics HL and further mathematics HL formula booklet** is required for this paper.

Name: _____

1.

[Maximum mark: 9]

Find the exact value of $\int_0^{\infty} \frac{\mathrm{d}x}{(x+2)(2x+1)}$.

2.

[Maximum mark: 11]

Find

(a) $\lim_{x\to 0} \frac{\tan x}{x+x^2}$; *[4 marks]*

(b) $\lim_{x\to 1} \frac{1-x^2+2x^2\ln x}{1-\sin\frac{\pi x}{2}}$. *[7 marks]*

3.

[Maximum mark: 12]

(a) Determine whether the series $\sum_{n=1}^{\infty} \sin\frac{1}{n}$ is convergent or divergent. *[5 marks]*

(b) Show that the series $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ is convergent. *[7 marks]*

MATHEMATICS
HIGHER LEVEL
PAPER 3 – STATISTICS AND PROBABILITY

31 marks

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Name: _____

1.

[Maximum mark: 11]

The weights of the oranges produced by a farm may be assumed to be normally distributed with mean 205 grams and standard deviation 10 grams.

(a) Find the probability that a randomly chosen orange weighs more than 200 grams. *[2 marks]*

(b) Five of these oranges are selected at random to be put into a bag. Find the probability that the combined weight of the five oranges is less than 1 kilogram. *[4 marks]*

(c) The farm also produces lemons whose weights may be assumed to be normally distributed with mean 75 grams and standard deviation 3 grams. Find the probability that the weight of a randomly chosen orange is more than three times the weight of a randomly chosen lemon. *[5 marks]*

2.

[Maximum mark: 12]

(a) The random variable X has the negative binomial distribution $\mathrm{NB}(3, p)$. Let $f(x)$ denote the probability that X takes the value x.

(i) Write down an expression for $f(x)$, and show that

$$\ln f(x) = 3\ln\left(\frac{p}{1-p}\right) + \ln(x-1) + \ln(x-2) + x\ln(1-p) - \ln 2 .$$

(ii) State the domain of f.

(iii) The domain of f is extended to $]2, \infty[$. Show that

$$\frac{f'(x)}{f(x)} = \frac{1}{x-1} + \frac{1}{x-2} + \ln(1-p) .$$ *[7 marks]*

(b) Jo has a biased coin which has a probability of 0.35 of showing heads when tossed. She tosses this coin successively and the 3rd head occurs on the Y^{th} toss. Use the result in part (a)(iii) to find the most likely value of Y. *[5 marks]*

3.

[Maximum mark: 8]

A shop sells apples, pears and peaches. The weights, in grams, of these three types of fruit may be assumed to be normally distributed with means and standard deviations as given in the following table.

Fruit	Mean	Standard Deviation
Apples	115	5
Pears	110	4
Peaches	105	3

Alan buys 1 apple and 1 pear while Brian buys 1 peach. Calculate the probability that the combined weight of Alan's apple and pear is greater than twice the weight of Brian's peach.

Solutions

1.

(a) $z = \frac{200-205}{10} = -0.5$ ***(M1)***

probability = 0.691 (accept 0.692) ***A1***

Note: Award ***M1A0*** for 0.309 or 0.308

[2 marks]

(b) let X be the total weight of the 5 oranges

then $\mathrm{E}(X) = 5\times 205 = 1025$ ***(A1)***

$\mathrm{Var}(X) = 5\times 100 = 500$ ***(M1)(A1)***

$\mathrm{P}(X < 1000) = 0.132$ ***A1***

[4 marks]

(c) let $Y = B - 3C$ where B is the weight of a random orange and C the weight of a random lemon ***(M1)***

$\mathrm{E}(Y) = 205 - 3\times 75 = -20$ ***(A1)***

$\mathrm{Var}(Y) = 100 + 9\times 9 = 181$ ***(M1)(A1)***

$\mathrm{P}(Y > 0) = 0.0686$ ***A1***

[5 marks]

Note: Award A1 for 0.0681 obtained from tables

Total [11 marks]

2.

(a) (i) $f(x)=\binom{x-1}{2}p^3(1-p)^{x-3}$ ***M1A1***

Note: Award ***M1A0*** for $f(x)=\binom{x-1}{2}p^3q^{x-3}$

taking logs, ***M1***

$$\ln f(x)=\ln\left(\binom{x-1}{2}p^3(1-p)^{x-3}\right)$$

$$=\ln\left(\frac{(x-1)(x-2)}{2}\times p^3(1-p)^{x-3}\right)$$ ***A1***

Note: Award ***A1*** for simplifying binomial coefficient, seen anywhere.

$$=\ln\left(\frac{(x-1)(x-2)}{2}\times p^3\frac{(1-p)^x}{(1-p)^3}\right)$$ ***A1***

Note: Award ***A1*** for correctly splitting $(1-p)^{x-3}$, seen anywhere.

$$=3\ln\left(\frac{p}{1-p}\right)+\ln(x-1)+\ln(x-2)+x\ln(1-p)-\ln 2$$ ***AG***

(ii) the domain is $\{3, 4, 5, \ldots\}$ ***A1***

Note: Do not accept $x\geq 3$

(iii) differentiating with respect to x, ***M1***

$$\frac{f'(x)}{f(x)}=\frac{1}{x-1}+\frac{1}{x-2}+\ln(1-p)$$ ***AG***

[7 marks]

(b) setting $f'(x)=0$ and putting $p=0.35$,

$$\frac{1}{x-1}+\frac{1}{x-2}+\ln 0.65=0$$ ***M1A1***

solving, $x=6.195\ldots$ ***A1***

we need to check $x=6$ and 7

$f(6)=0.1177\ldots$ and $f(7)=0.1148\ldots$ ***A1***

the most likely value of Y is 6 ***A1***

Note: Award the final ***A1*** for the correct conclusion even if the previous ***A1*** was not awarded.

[5 marks]

Total [12 marks]

3.

let X, Y, Z denote respectively the weights, in grams, of a randomly chosen apple, pear, peach

then $U = X + Y - 2Z$ is $\mathrm{N}(115 + 110 - 2\times 105, 5^2 + 4^2 + 2^2 \times 3^2)$ ***(M1)(A1)(A1)***

Note: Award ***M1*** for attempted use of U.

i.e. $\mathrm{N}(15, 77)$ ***A1***

we require

$\mathrm{P}(X + Y > 2Z) = \mathrm{P}(U > 0)$ ***M1A1***

$= 0.956$ ***A2***

Note: Award ***M0A0A2*** for 0.956 only.

[8 marks]

4.