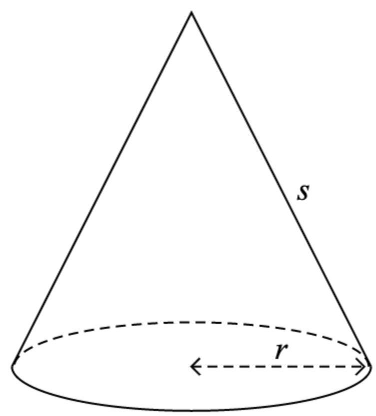
1. [Maximum points: 28]

In this problem you will investigate the area of surfaces generated by rotating functions 360° around the x-axis.

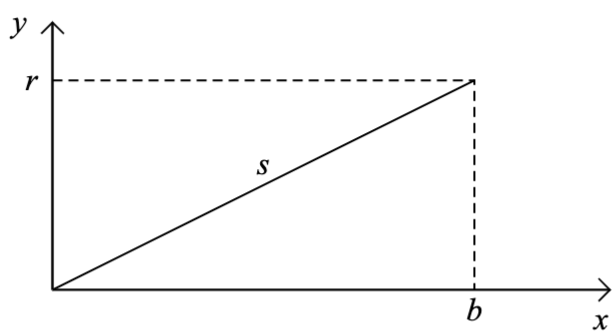
When the function $y = f(x)$ from $x = a$ to $x = b$ is rotated 360° around the x-axis the area A of the surface generated is given by

$$A = 2\pi\int_a^b f(x)\sqrt{1 + (f'(x))^2}\,dx$$

Consider a cone with a slant height of s and base of radius r as shown in the diagram below.



The curved surface is formed when the line with length s in the diagram below from $x = 0$ to $x = b$ is rotated 360° around the x-axis.



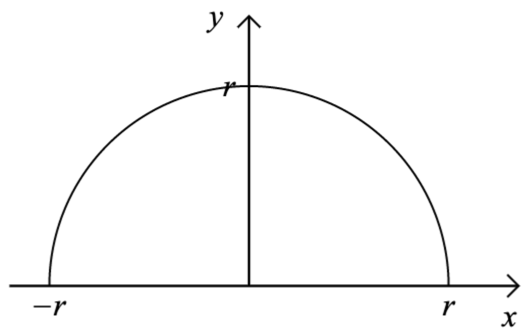
(a) In terms of s and r find [4]

(i) the value of b

(ii) the equation of the line

(b) Hence show that the area of the curved surface of the cone is equal to πrs. [5]

The diagram below shows a semi-circle of radius r centred at the origin. Let the equation of the semi-circle be $y = g(x)$.



(c) Find [5]

(i) $g(x)$

(ii) $g'(x)$

(d) Hence show that the surface area of a sphere is equal to $4\pi r^2$. [6]

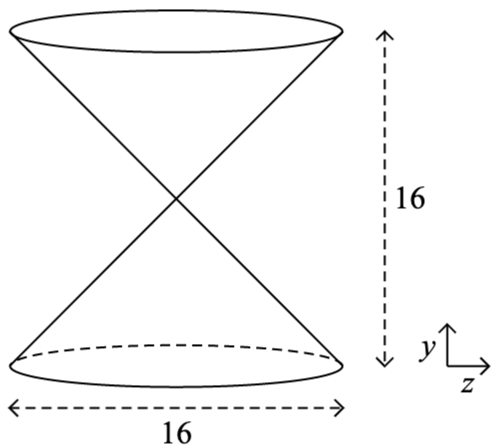
(e) Find the exact area of the surface generated when then function $y = x^3$ from $x = 1$ to $x = 2$ is rotated 360° around the x-axis. [8]

2. [Maximum points: 41]

In this problem you will investigate the shape formed by the intersection of a cone and a plane.

Two identical cones of height 8 with base of diameter 16 placed tip-to-tip with the centre of the top cone directly above the centre of the bottom cone.

This is shown in the diagram below where the directions of the z and y-axes are given, and the positive x-direction is out of the page.



Let the point where the tips meet have coordinates $(0,0,1)$.

Consider a point on the curved surface of a cone with coordinates (x,y,z).

(a) Show that $x^2 + (z-1)^2 = z^2$. [4]

The equation of a **plane** is $z = 0$.

(b) Find the equation of the curve created by the intersection of the plane $z = 0$ and the two cones. Write your answer in the form $ax^2 + by^2 = 1$ where a and b are real numbers to be determined. [2]

(c) For your equation in part (b) find [9]

(i) $\dfrac{dy}{dx}$

(ii) the coordinates of any turning points

(iii) the equations of any linear oblique asymptotes

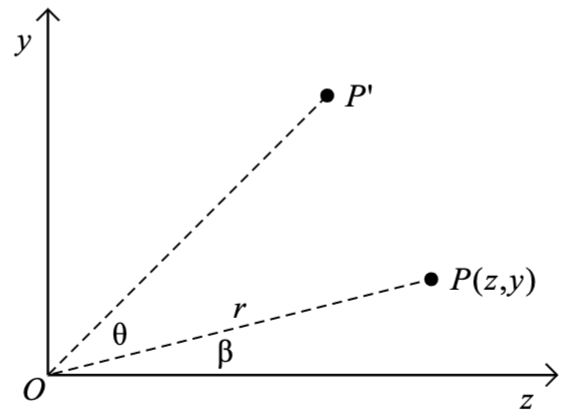
(d) Sketch the curve represented by the equation in part (b) for $-5 \le x \le 5$ showing the features from (c) parts (ii) and (iii). [3]

(e) Verify that the vector equation of the curved surface of the cones can be written as [3]

$$\boldsymbol{r} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} \cos t \\ 1 \\ \sin t \end{pmatrix}$$

where $\lambda, t \in \mathbb{R}$.

The diagram below shows point P in the zy-plane with coordinates (z,y) at a distance r from the origin O. The point is rotated by an angle of θ anti-clockwise about the origin to form point P'. The angle between OP and the z-axis is β.



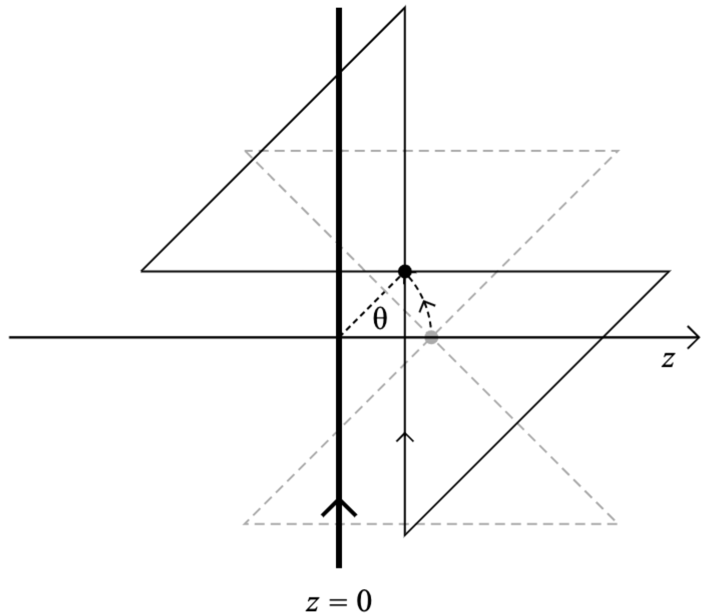
(f) Show that the coordinates of P' are [3]

$$(r\cos\theta\cos\beta - r\sin\theta\sin\beta, r\sin\theta\cos\beta + r\cos\theta\sin\beta)$$

(g) Hence show that this is equal to [2]

$$(z\cos\theta - y\sin\theta, z\sin\theta + y\cos\theta)$$

The cones are now rotated in the *zy*-plane about the origin so that the plane $z = 0$ is now parallel to one edge of each cone when viewed in the *zy*-plane. This is shown in the diagram below.



(h) Calculate the angle of rotation θ. [2]

(i) Show that the vector equation of the curved surface of the rotated cones is [4]

$$\boldsymbol{r} = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\1\end{pmatrix} + \frac{\lambda}{\sqrt{2}}\begin{pmatrix}\sqrt{2}\cos t\\ \sin t + 1\\ \sin t - 1\end{pmatrix}$$

(j) For any point on the curve created by the intersection of the plane $z = 0$ and the rotated cones show that [5]

(i) $y = \lambda\sqrt{2}$

(ii) $x^2 = 2\lambda - 1$

(k) Find the equation of this curve and sketch its graph for $-5 \le x \le 5$ showing the coordinates of any axes intercepts. [4]

1. (a)

(i) Use the Pythagorean theorem — M1

$$b = \sqrt{s^2 - r^2}$$ A1

(ii) $y = \dfrac{rx}{\sqrt{s^2 - r^2}}$ — A1A1

(b) Use the given formula — M1

$$A = 2\pi \int_0^{\sqrt{s^2 - r^2}} \frac{rx}{\sqrt{s^2 - r^2}} \cdot \sqrt{1 + \frac{r^2}{s^2 - r^s}}\, dx = 2\pi \int_0^{\sqrt{s^2 - r^2}} \frac{rsx}{s^2 - r^2}\, dx$$ A1

Evaluate the integral

$$A = \pi \left[\frac{rsx^2}{(s^2 - r^2)} \right]_0^{\sqrt{s^2 - r^2}} = \frac{\pi rs(s^2 - r^2)}{s^2 - r^2} = \pi rs$$ M1A1A1

(c)

(i) Use the Pythagorean theorem — M1

$$g(x) = \sqrt{r^2 - x^2}$$ A1

(ii) Use the chain rule — M1

$$g(x) = \frac{-2x}{2\sqrt{r^2 - x^2}} = -\frac{x}{\sqrt{r^2 - x^2}}$$ A1A1

(d) Use the given formula — M1

$$A = 2\pi \int_{-r}^{r} \sqrt{r^2 - x^2} \cdot \sqrt{1 + \frac{x^2}{r^2 - x^2}}\, dx = 2\pi \int_{-r}^{r} \frac{r\sqrt{r^2 - x^2}}{\sqrt{r^2 - x^2}}\, dx = 2\pi \int_{-r}^{r} r\, dx$$ A1A1

Evaluate the integral

$$A = 2\pi [rx]_{-r}^{r} = 2\pi\big(r^2 - (-r^2)\big) = 4\pi r^2$$ M1A1A1

(e) We have

$$\frac{dy}{dx} = 3x^2$$ A1

Use the given formula M1

$$A = 2\pi\int_0^2 x^3\sqrt{1 + 9x^4}\, dx$$ A1

Use the substitution $u = 1 + 9x^4$. So $\frac{du}{dx} = 36x^3$. M1

When $x = 0$ then $u = 1$ and when $x = 2$ then $u = 145$. A1

The integral then becomes

$$A = \frac{\pi}{18}\int_1^{145} u^{1/2}\, du = \frac{\pi}{18}\left[\frac{2u^{3/2}}{3}\right]_1^{145} = \frac{\pi(145^{3/2} - 1)}{27}$$ M1A1A1

2. (a) Consider the vertical line passing through the centres of the cones.

The distance of any point on the curved surface to this line will always be

$$\sqrt{x^2 + (z-1)^2} \quad \text{A1}$$

The y-coordinate will always be equal this value. R1

So

$$y = \sqrt{x^2 + (z-1)^2} \quad \text{M1}$$

Rearranging gives

$$x^2 + (z-1)^2 = y^2 \quad \text{A1}$$

(b) We have

$$x^2 + (0-1)^2 = y^2 \quad \text{M1}$$

So

$$y^2 - x^2 = 1 \quad \text{A1}$$

(c)

(i) Use implicit differentiation M1

$$2y \cdot \frac{dy}{dx} - 2x = 0 \quad \text{A1}$$

So

$$\frac{dy}{dx} = \frac{x}{y} \quad \text{A1}$$

(ii) We have $\frac{dy}{dx} = 0$ giving $x = 0$. So A1

$$y^2 = 1 \quad \text{M1}$$

Giving $y = \pm 1$. A1

The coordinates are therefore $(0, \pm 1)$.

(iii) Rearrange

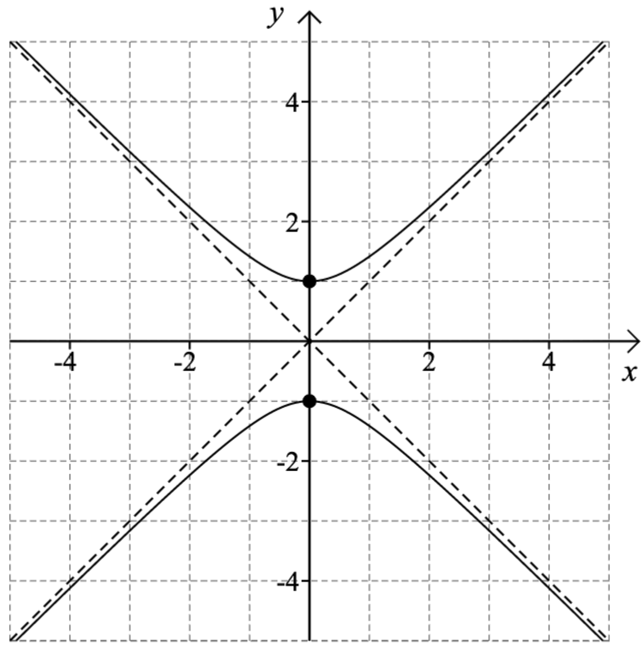
$$y = \pm\sqrt{x^2+1} = \pm x\sqrt{1 + \frac{1}{x^2}} \quad \text{M1A1}$$

So the equations are $y = \pm x$. A1

(d) Draw the asymptotes. A1

Label the points (0,1) and (0,-1). A1

Draw two symmetrical curves passing through these points and approaching the asymptotes. A1



(e) We have

$$x = \lambda \cos t$$

$$y = \lambda$$

$$z = 1 + \lambda \sin t$$ A1

Substitute these into the equation from (a). M1

$$\lambda^2 \cos^2 t + \lambda^2 \sin^2 t = \lambda^2$$

This simplifies to $\lambda^2 = \lambda^2$ which is correct. A1

(f) The coordinates are

$$\left(r\cos(\theta + \beta), r\sin(\theta + \beta)\right)$$ A1

Use the double angle formulae. This gives M1

$$(r\cos\theta\cos\beta - r\sin\theta\sin\beta, r\sin\theta\cos\beta + r\cos\theta\sin\beta)$$ A1

(g) Since $z = r\cos\beta$ and $y = r\sin\beta$ the coordinates become M1

$$(z\cos\theta - y\sin\theta, z\sin\theta + y\cos\theta)$$ A1

(h) $\arctan(1/1) = 45°$ M1A1

(i) Let the new coordinates be (x', y', z').

We have

$$x' = \lambda\cos t$$ A1

$$y' = (1 + \lambda\sin t)\frac{\sqrt{2}}{2} + \lambda\frac{\sqrt{2}}{2}$$ A1

$$z' = (1 + \lambda\sin t)\frac{\sqrt{2}}{2} - \lambda\frac{\sqrt{2}}{2}$$ A1

The vector equation is therefore

$$\boldsymbol{r} = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\1\end{pmatrix} + \frac{\lambda}{\sqrt{2}}\begin{pmatrix}\sqrt{2}\cos t\\ \sin t + 1\\ \sin t - 1\end{pmatrix}$$ A1

(j)

(i) We have

$$y = \frac{1}{\sqrt{2}} + \frac{\lambda}{\sqrt{2}}(\sin t + 1)$$

and

$$0 = \frac{1}{\sqrt{2}} + \frac{\lambda}{\sqrt{2}}(\sin t - 1)$$

Subtracting gives M1

$$y = \lambda\sqrt{2}$$ A1

(ii) We have

$$x = \lambda\cos t$$

and

$$0 = \frac{1}{\sqrt{2}} + \frac{\lambda}{\sqrt{2}}(\sin t - 1)$$

So

$$\lambda\sin t = \lambda - 1$$ A1

Therefore

$$(\lambda - 1)^2 + x^2 = \lambda^2$$ M1

Giving

$$x^2 = 2\lambda - 1$$ A1

(k) We have

$$\lambda = \frac{x^2 + 1}{2}$$ M1

So

$$y = \frac{x^2 + 1}{\sqrt{2}}$$ A1

Draw a parabola that is concave upwards. A1

The y-intercept is $1/\sqrt{2}$. A1