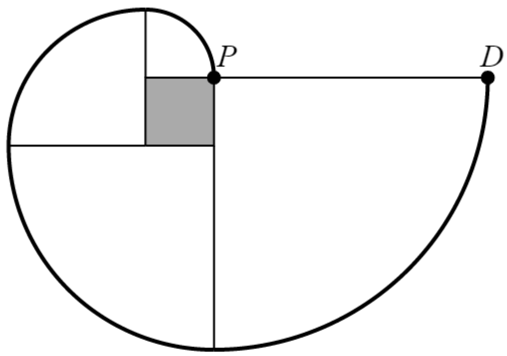
1. [Maximum points: 28]

A dog is attached to the top-right corner of a square building of perimeter 8 m with a piece of rope also of length 8 m. The initial position of the dog D is such that it is collinear with the top two corners of the building.

The dog walks clockwise around the building, keeping the rope tight at all times, until he can do so no more.

This is shown in the diagram below.



(a) The dog walks through four different arcs. Determine [6]

(i) the angle subtended by each arc

(ii) the radius of each arc in descending order

(iii) the total distance the dog walks.

The building is now in the shape of a regular octagon with perimeter 8 m. The dog walks clockwise around the building, keeping the rope tight at all times, until he can do so no more.

This is shown in the diagram below.

(b) The dog walks through eight different arcs. Determine [6]

(i) the angle subtended by each arc

(ii) the radius of each arc in descending order

(iii) the total distance the dog walks

The dog is now tied to a tree with a trunk of circumference 8 m. The initial position of the dog is such that the rope is tangential to the top of the trunk. The dog walks clockwise around the tree, keeping the rope tight at all times, until he can do so no more.

This is shown in the diagram below.

(c) By considering a regular n-gon as $n \to \infty$, show that the distance the dog walks around the tree is equal to 8π metres. [11]

(d) After reaching point P on the tree the dog decides to walk anti-clockwise, keeping the rope tight at all times, until he can do so no more. Determine the distance that he walks, excluding the initial distance calculated in part (c). [5]

2. [Maximum points: 27]

In this problem you will investigate properties of hyperbolic functions and use them to evaluate a definite integral.

Let the *hyperbolic* functions $\sin hx$ and $\cos hx$ be defined as

$$\sin hx = \frac{e^x - e^{-x}}{2}$$

and

$$\cos hx = \frac{e^x + e^{-x}}{2}$$

(a) Show that [11]

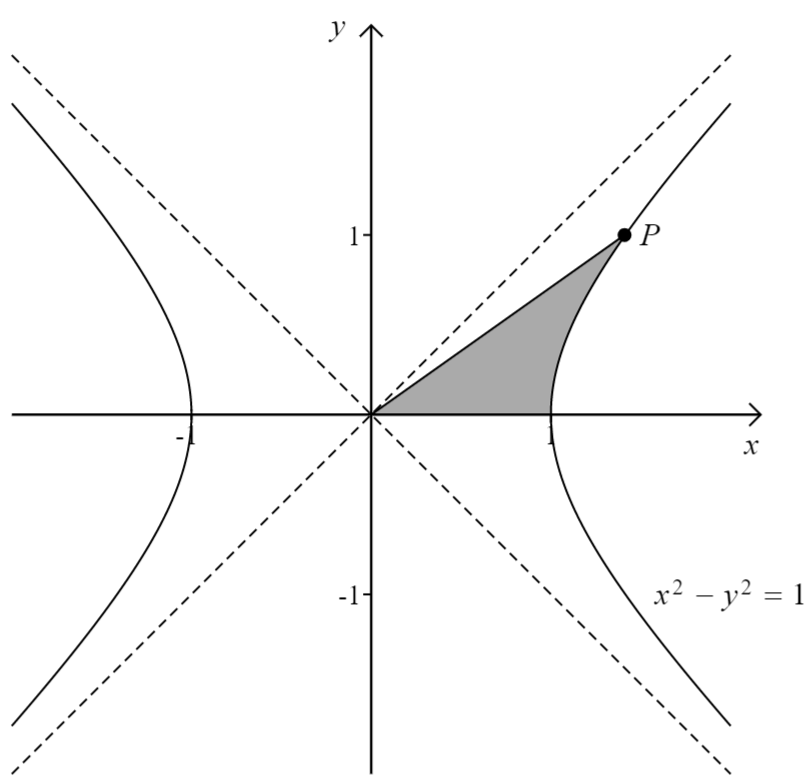
(i) $\dfrac{d}{dx}(\sin hx) = \cos hx$

(ii) $\dfrac{d}{dx}(\cos hx) = \sin hx$

(iii) $1 + \sin h^2 x = \cos h^2 x$

(iv) $\int \cos h^2 x = \dfrac{x + \sin hx \cos hx}{2} + C$

A certain *hyperbola* has the equation $x^2 - y^2 = 1$. Its graph is shown below. A line connects point P on the hyperbola to the origin. The area between this line, the hyperbola and the x-axis is shaded.



(b) Find the equations of the linear oblique asymptotes. [5]

(c) If point P has an x-coordinate of $\cosh a$, where $a \in \mathbb{R}$, in terms of a find [6]

(i) the y-coordinate of point P

(ii) an integral that represents the area of the shaded region (do not evaluate this integral)

(d) Use the substitution $y = \sinh u$ to evaluate the integral from (c) part (ii). [5]

1. (a)

(i) $\frac{360}{4} = 90°$ M1A1

(ii) The square has sides of length 2 m. R1

So the radii are 8 m, 6 m, 4 m and 2 m. A1

(iii) Use the arc length formula to determine the total distance the dog walks. M1

$\frac{90}{360} \times 2\pi \times 8 + \frac{90}{360} \times 2\pi \times 6 + \frac{90}{360} \times 2\pi \times 4 + \frac{90}{360} \times 2\pi \times 2 = 10\pi$ A1

(b)

(i) $\frac{360}{8} = 45°$ M1A1

(ii) The octagon has sides of length 1 m. R1

So the radii are 8 m, 7 m, 6 m, 5 m, 4 m, 3 m, 2 m and 1 m. A1

(iii) Use the arc length formula to determine the total distance the dog walks. M1

$$\frac{45}{360} \times 2\pi \times \sum_{k=1}^{8} k = 9\pi$$ A1

(c) If we consider a regular n-gon then as $n \to \infty$ the shape gets closer to being a circle. R1

The angle subtended by each arc is equal to $\frac{360}{n}$. A1

The regular n-gon has sides of length $\frac{8}{n}$. A1

Use the arc length formula to determine the total distance the dog walks. M1

$$\frac{360}{n} \div 360 \times 2\pi \times \sum_{k=1}^{n} 8 - (k-1) \times \frac{8}{n}$$ A1

The sigma notation is an arithmetic series with $t_1 = 8$ and $d = -\frac{8}{n}$. R1

Use the arithmetic series formula to evaluate the sigma notation. M1

$$\frac{2\pi}{n} \times \frac{n}{2} \times \left[2 \times 8 + (n-1) \times \left(-\frac{8}{n}\right)\right]$$

This simplifies to

$$\pi\left(16 - \frac{8(n-1)}{n}\right)$$ A1

Consider the limit as $n \to \infty$. M1

$$\lim_{n \to \infty} \pi\left(16 - \frac{8(n-1)}{n}\right) = \lim_{n \to \infty} \pi\left(16 - \frac{8\left(1 - \frac{1}{n}\right)}{1}\right)$$ A1

This is equal to 8π. A1

(d) The distance back to point D is 8π. A1

The dog then walks through a semi-circle of radius 8 metres. This distance is equal to

$$\frac{1}{2} \times 2\pi \times 8 = 8\pi$$ M1A1

The dog can then walk another 8π metres around the tree until he arrives at point P again. A1

So altogether he walks 24π metres. He is a good boy. A1

2. (a)

(i) $\frac{d}{dx}(\sinh x) = \frac{e^x - (-e^{-x})}{2} = \frac{e^x + e^{-x}}{2} = \cosh x$ M1A1

(ii) $\frac{d}{dx}(\cosh x) = \frac{e^x + (-1)e^{-x}}{2} = \frac{e^x - e^{-x}}{2} = \sinh x$ M1A1

(iii) We have

$$1 + \sinh^2 x = 1 + \frac{e^{2x} - 2 + e^{-2x}}{4} = \frac{e^{2x} + 2 + e^{-2x}}{4} = \cosh^2 x$$ M1A1A1

(iv) We have

$$\int \frac{e^{2x} + 2 + e^{-2x}}{4} dx = \frac{x}{2} + \frac{e^{2x} + e^{-2x}}{8} = \frac{x}{2} + \frac{1}{2} \cdot \frac{e^x - e^{-x}}{2} \cdot \frac{e^x + e^{-x}}{2}$$ M1A1A1

This simplifies to

$$\frac{x + \sinh x \cosh x}{2} + C$$ A1

(b) We have

$$y^2 = x^2 - 1$$ A1

So

$$y = \pm\sqrt{x^2 - 1} = \pm x\sqrt{1 - \frac{1}{x^2}}$$ A1A1

Since

$$\lim_{x \to \pm\infty} \pm x\sqrt{1 - \frac{1}{x^2}} = \pm x$$ M1

The asymptotes are $y = \pm x$. A1

(c)

(i) We have

$$\cosh^2 x - y^2 = 1$$ M1

So

$$y^2 = \cosh^2 x - 1 = \sinh^2 x$$ A1A1

Giving $y = \sinh x$. A1

(ii) $\int_o^{\sinh a} \sqrt{1 + y^2} - \frac{\cosh a}{\sinh a} y \, dy$ A1A1

(d) We have $y = \sinh u$ so

$$\frac{dy}{du} = \cosh u$$ A1

The integral then becomes

$$\int_0^a \cosh^2 u \, du - \frac{\cosh a}{\sinh a}\int_0^{\sinh a} y \, dy$$ M1

This is equal to

$$\left[\frac{y + \sinh y \cosh y}{2}\right]_0^a - \frac{\cosh a}{\sinh a}\left[\frac{y^2}{2}\right]_0^{\sinh a} = \frac{a + \sinh a \cosh a}{2} - \frac{\sinh a \cosh a}{2}$$ A1A1

This simplifies to $\frac{a}{2}$. A1